Final exam for Kwantumfysica 1 - 2011-2012
Thursday 9 February 2012, 13:00 - 16:00

READ THIS FIRST:

- Note that the lower half of this page lists some useful formulas and constants.
- Clearly write your name and study number on each answer sheet that you use.
- On the first answer sheet, write the total number of answer sheets that you turn in.
- The exam has several questions, it continues on the backside of the papers.
- Start each question (number 1, 2, etc.) on a new answer sheet.
- The exam is open book with limits. You are allowed to use the book by Griffiths or Liboff, the handouts *Extra note on two-level systems and exchange degeneracy for identical particles*, and *Feynman Lectures chapter III-1*, one A4 sheet with your own notes, but nothing more than this.
- If it says "make a rough estimate", there is no need to make a detailed calculation, and making a simple estimate is good enough. If it says "calculate" or "derive", you are supposed to present a full analytical calculation.
- If you get stuck on some part of a problem for a long time, it may be wise to skip it and try the next part of a problem first.

Useful formulas and constants:

Electron mass	m_e	$= 9.1 \cdot 10^{-31}$ kg
Electron charge	$-e$	$= -1.6 \cdot 10^{-19}$ C
Planck's constant	h	$= 6.626 \cdot 10^{-34}$ Js $= 4.136 \cdot 10^{-15}$ eVs
Planck's reduced constant	$\hbar$	$= 1.055 \cdot 10^{-34}$ Js $= 6.582 \cdot 10^{-16}$ eVs

Fourier relation between x-representation and k-representation of a state

$$\Psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{\Psi}(k) e^{ikx}\, dk$$

$$\overline{\Psi}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x) e^{-ikx}\, dx$$

Problem 1

a) One measures the length of the *orbital* angular momentum vector for an electron in a hydrogen atom. The measurement outcome is $\sqrt{20}\,\hbar$. Next, one plans to measures the z-component of the orbital angular momentum of this electron. What are the possible measurement outcomes?

b) Consider an electron at rest in a space without electric and magnetic fields. One measures the z-component of the spin of the electron. The measurement outcome is spin-up along the z-axis. Next, one plans to measure the x-component of the spin of the electron.
What are the possible measurement outcomes? And what is the probability of each outcome for this second measurement?

Problem 2
The position x of a particle is at some time $t = 0$ described by the normalized, real-valued wavefunction

$$\Psi(x) = Ae^{-|x/b|} \quad ,$$

with $b = 1$ nm.

a) Make a sketch of both $\Psi(x)$ and the probability density $W(x)$ for the particle's position.

b) Show that the state is normalized for $A = 1$ nm$^{-1/2}$, and explain the unit of A.

c) Write down the expression for the expectation value $\langle \hat{x} \rangle$ for this state, and evaluate the answer.
(If needed for your approach, you could use $\int x \cdot e^{cx} dx = \left(cx \cdot e^{cx} - e^{cx}\right)/c^2$.)

d) Explain how you could get the answer for **c)** without doing the full calculation.

e) What is the expectation value $\langle \hat{p}_x \rangle$ for the particle's momentum at time $t = 0$? Support the answer by showing a calculation, also when you can guess the answer.

f) With the particle's wavefunction as sketched, you plan to measure the position x. What is the probability for detecting a value in the range 0 nm $< x <$ 1 nm?

g) You measure the position x, with a measurement apparatus that has a resolution of 0.1 nm. You detect the particle at the position $x = 1.5$ nm. Make a sketch of the probability density $W(x)$ for the particle's position, immediately after the measurement. Explain your answer, and the width, height and area of $W(x)$ in your sketch.

h) Assume that the particle is again in the original state $\Psi(x)$ (as above **a)**). For studying the momentum (and velocity) properties of the particle in it is useful to describe the state of the particle with a wavefunction $\overline{\Psi}(k)$ as a function of wavenumber k. Here k is related to the momentum p according to $p = \hbar k$.
Make a sketch of $\overline{\Psi}(k)$ after first calculating $\overline{\Psi}(k)$.

Problem 3
Consider the following model system for an atom with one electron: a one-dimensional particle-in-a-box system, where the potential for the electron outside the box is infinite, and inside the box the potential $V = 0$. The position of the electron is described by a coordinate x. The width of the box is a, with the walls at $x = -a/2$ and $x = +a/2$. The eigenvalues for total energy of this system are denoted as E_n, and belong to energy eigenstates $\varphi_n(x)$ which are

$$\varphi_n(x) = \sqrt{\tfrac{2}{a}} \cos\left(\tfrac{n\pi x}{a}\right), \qquad \text{for } n = 1, 3, 5, 7, \ldots$$
$$\varphi_n(x) = \sqrt{\tfrac{2}{a}} \sin\left(\tfrac{n\pi x}{a}\right), \qquad \text{for } n = 2, 4, 6, 8, \ldots$$

Assume that this system has an electrical dipole moment that oscillates when the system is emitting a photon. This can occur when the system is in a superposition of two different energy eigenstates $|\varphi_m\rangle$ and $|\varphi_n\rangle$. The operator for this dipole moment is $\hat{D} = e\hat{X}$, where $\hat{X}$ the position operator and e the electron charge.

We introduce here the parity operator $\hat{P}$, which is defined by how it works on a function $f(x)$:

$$\hat{P}f(x) = f(-x).$$

It can be shown that $\hat{P}$ has two eigenvalues +1 (*even parity*) and -1 (*odd parity*). Any even function is an eigenfunction for the eigenvalue +1, while any odd function is an eigenfunction for the eigenvalue -1. $\hat{P}$ can also be used to characterize the symmetry (parity) of operators.

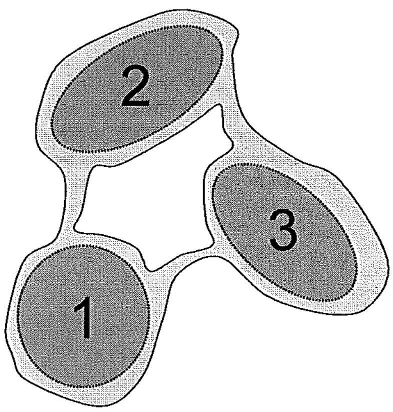
a) Evaluate $\hat{P}\,\varphi_n(x)$ and derive for which n the state $\varphi_n(x)$ has even or odd parity.

b) What is the parity of $\hat{D}$? **Hint:** analyze it in the x-representation.

c) Use symmetry and parity arguments to show that this particle-in-a-box system cannot emit a photon when it is in a state that is a superposition of two energy eigenstates with the same parity.
Hint: use the x-representation to evaluate all the elements like $\langle\varphi_n|\hat{D}|\varphi_m\rangle$ in an expression for how the dipole moment oscillates as a function of time during the emission of a photon. If you want you can assume that $\langle\varphi_n|\hat{D}|\varphi_m\rangle$ is real and that $\langle\varphi_n|\hat{D}|\varphi_m\rangle = \langle\varphi_m|\hat{D}|\varphi_n\rangle$.

d) An electron is in the third excited state of this system (from the four lowest energy eigenstates, the one with the highest energy). It can (and will) relax to lower energy eigenstates by spontaneous emission of a photon during the transition to this lower state. Discuss which relaxation processes are possible, and for each which photon is (or photons are) emitted, and what the final state is.

Problem 4 FOR THIS PROBLEM YOU MUST USE DIRAC NOTATION.
An electron in a certain organic molecule can be located at three different positions in the molecule. The molecule is ring shaped, but its shape is not very symmetric. See also the following sketch.



When the electron is located in one of these positions 1, 2 or 3, its state will be denoted as $|x_1\rangle, |x_2\rangle$ or $|x_3\rangle$.

One can investigate a single molecule of this type by positioning it in a nanodevice that contains a single molecule in the focus of a laser beam. By shining a short pulse from a blue laser (light of 468 nm) on this device and looking at the scattered light, one can measure in which of the positions 1, 2 or 3 the electron is located. That is, with the blue laser pulse one can perform a position measurement.
Alternatively, one can do a measurement with a short pulse from a red laser (light of 745 nm). Now the scattered light is used for measuring the amount of vibrational energy A in the molecule. The observable $\hat{A}$ for this property has 3 eigenvalues, that obey the following equations:

$$\hat{A}|\alpha_1\rangle = A_1\,|\alpha_1\rangle, \quad \text{where } |\alpha_1\rangle = -\sqrt{\frac{1}{2}}|x_1\rangle + \sqrt{\frac{1}{2}}|x_2\rangle$$

$$\hat{A}|\alpha_2\rangle = A_2\,|\alpha_2\rangle, \quad \text{where } |\alpha_2\rangle = -\sqrt{\frac{1}{6}}|x_1\rangle - \sqrt{\frac{1}{6}}|x_2\rangle + \sqrt{\frac{2}{3}}|x_3\rangle$$

$$\hat{A}|\alpha_3\rangle = A_3\,|\alpha_3\rangle, \quad \text{where } |\alpha_3\rangle = +\sqrt{\frac{1}{3}}|x_1\rangle + \sqrt{\frac{1}{3}}|x_2\rangle + \sqrt{\frac{1}{3}}|x_3\rangle$$

Before starting one or more measurements, the molecule is always prepared in the following state, simply by waiting a long time.

$$|\Psi_0\rangle = -\sqrt{\frac{2}{3}}|x_1\rangle + \sqrt{\frac{1}{6}}|x_2\rangle + \sqrt{\frac{1}{6}}|x_3\rangle$$

a) After preparing the system in the state $|\Psi_0\rangle$, a pulse from the red laser is applied to measure A. What are the possible measurement outcomes, and what is the probability for each of these outcomes?

b) After preparing the system in the state $|\Psi_0\rangle$, one applies a blue pulse and immediately after it a red pulse. The blue pulse did perform a measurement, with the result that the electron is at position 3. What is now the probability that the result from the measurement with the red pulse gives as result $A = 4$?

c) After preparing the system in the state $|\Psi_0\rangle$, one applies a pulse from the blue laser two times on a row. What is the probability to find a certain position result with the first pulse, and to find the electron in a neighboring position (with respect to the first result) on the ring as the result from the second pulse?

d) After preparing the system in the state $|\Psi_0\rangle$, what is the expectation value for A?